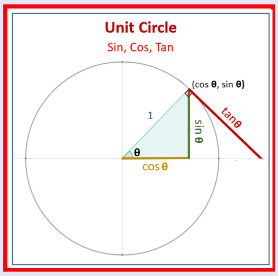
Math 241

Winter 2023

Lecture 15

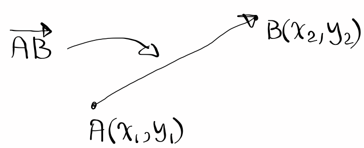


Feb 19-8:47 AM

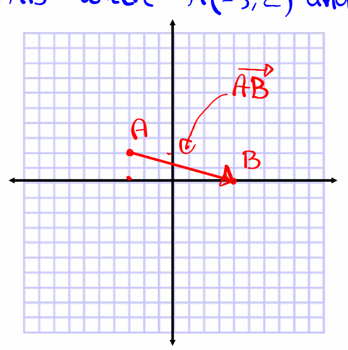
Introduction to Vectors:

Vectors are simply a directed line segment. It has an initial point and terminal point.

Ex: $A(x_1, y_1)$, $B(x_2, y_2)$



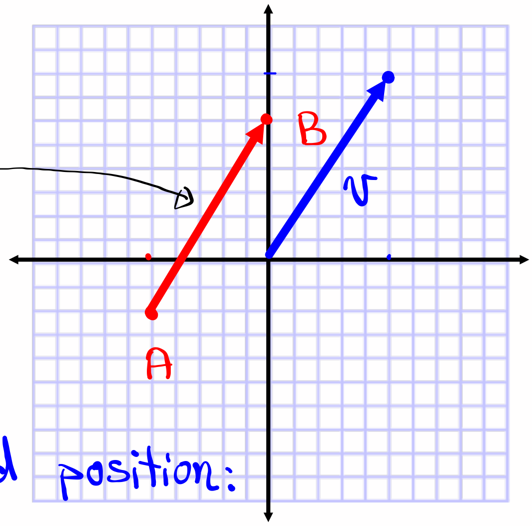
Draw vector $\vec{v} = \overrightarrow{AB}$ where $A(-3, 2)$ and $B(4, 0)$.



Jan 30-7:10 AM

Draw $\overrightarrow{AB}$ with $A(-5,-2)$ & $B(0,6)$.

$v = \langle x_2 - x_1 \, , \, y_2 - y_1 \rangle$



$v = \langle 0-(-5), 6-(-2) \rangle$

$= \langle 5, 8 \rangle$

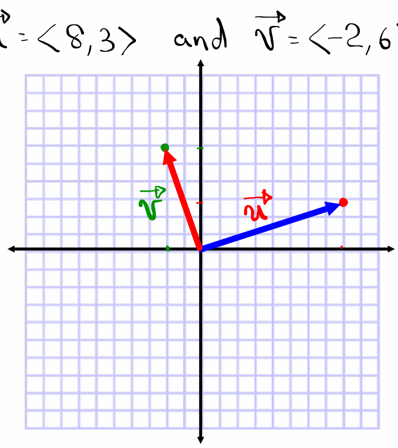
Vectors in standard position:

Initial Point at $(0,0)$

and Terminal Point at $(x_2 - x_1 \, , \, y_2 - y_1)$

Jan 30-7:14 AM

Draw $\vec{u} = \langle 8,3 \rangle$ and $\vec{v} = \langle -2,6 \rangle$

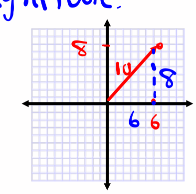


Consider vector $v = \langle a, b \rangle$

its length or magnitude $|v| = \sqrt{a^2 + b^2}$

Suppose $v = \langle 6, 8 \rangle$, find its magnitude.

$|v| = \sqrt{a^2 + b^2} = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$



Jan 30-7:19 AM

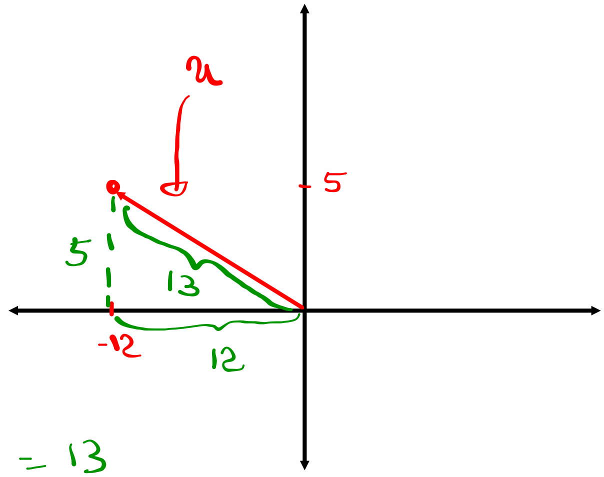
Given $u = \langle -12, 5 \rangle$

1) Draw u

2) $|u|$

$= \sqrt{(-12)^2 + 5^2} = \sqrt{144+25}$

$= \sqrt{169} = 13$



Jan 30-7:25 AM

Some operations with Vectors:

$u = \langle a_1, b_1 \rangle$ & $v = \langle a_2, b_2 \rangle$

1) $u + v = \langle a_1 + a_2, b_1 + b_2 \rangle$

2) $u - v = \langle a_1 - a_2, b_1 - b_2 \rangle$

3) $u \cdot v = a_1 a_2 + b_1 b_2$

dot Product

4) $cu = c\langle a_1, b_1 \rangle = \langle ca_1, cb_1 \rangle$ c is a real #.

Jan 30-7:28 AM

$u=\langle 2,-3\rangle$, $v=\langle -1,2\rangle$

1) $u+v=\langle 2+(-1), -3+2\rangle = \langle 1,-1\rangle$

2) $u-v=\langle 2-(-1), -3-2\rangle = \langle 3,-5\rangle$

3) $u\cdot v = 2(-1)+(-3)(2) = -2-6 = \boxed{-8}$

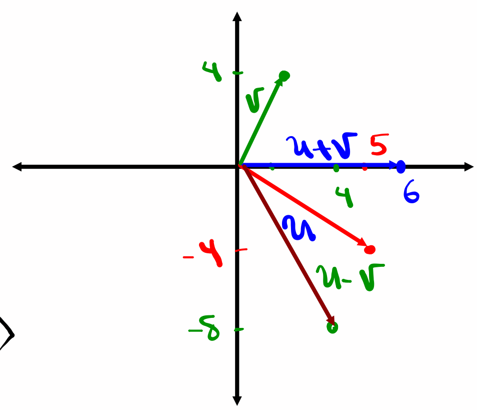
4) $4u = 4\langle 2,-3\rangle = \langle 8,-12\rangle$

5) $-2v = -2\langle -1,2\rangle = \langle 2,-4\rangle$

Jan 30-7:32 AM

$u=\langle 5,-4\rangle$, $v=\langle 1,4\rangle$

1) Draw u & v.



2) Find and draw $u+v$.

$u+v=\langle 5+1, -4+4\rangle = \langle 6,0\rangle$

3) Find and draw $u-v$

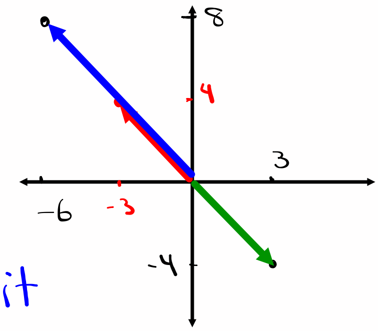
$u-v=\langle 5-1, -4-4\rangle = \langle 4,-8\rangle$

4) Find $u\cdot v = 5(1)+(-4)(4) = 5-16 = \boxed{-11}$

"u dot v"

Jan 30-7:36 AM

Given $u = \langle -3, 4 \rangle$



1) Draw u and find $|u|$

$|u| = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$

2) Find $2u$, and draw it

$2u = 2\langle -3, 4 \rangle = \langle -6, 8 \rangle$ $\quad |2u| = \sqrt{(-6)^2 + 8^2} = 10$

3) Find $-u$, and draw it

$-u = -1\langle -3, 4 \rangle = \langle 3, -4 \rangle$

4) Find $u \cdot u = \langle -3, 4 \rangle \cdot \langle -3, 4 \rangle = (-3)(-3) + 4(4) = 25$

So $u \cdot u = |u|^2$

Jan 30-7:43 AM

Special $\underbrace{\text{Unit Vectors}}_{\text{length} = 1}$:

$i = \langle 1, 0 \rangle$, $j = \langle 0, 1 \rangle$

along x-axis along y-axis

$v = \langle a, b \rangle = a\langle 1, 0 \rangle + b\langle 0, 1 \rangle$

$= ai + bj$

$v = -4i - 5j = -4\langle 1, 0 \rangle - 5\langle 0, 1 \rangle$

$= \langle -4, 0 \rangle + \langle 0, -5 \rangle$

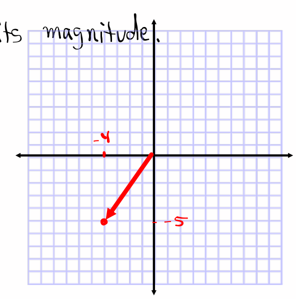
$= \langle -4, -5 \rangle$

Draw v, and find its magnitude.

$|u| = \sqrt{(-4)^2 + (-5)^2}$

$= \sqrt{16 + 25}$

$= \sqrt{41}$

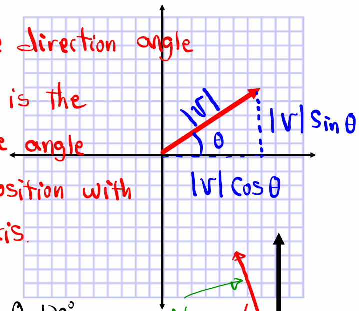


Jan 30-7:52 AM

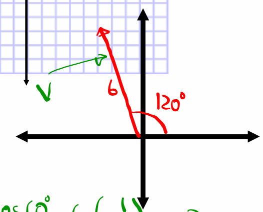
Consider the vector v below:

θ is called the direction angle

Direction angle is the smallest positive angle in standard position with positive x-axis.



Suppose $|v| = 6$, $\theta = 120°$



Horizontal Component:

$|v|\cos\theta = 6 \cdot \cos 120° = 6 \cdot -\cos 60° = 6 \cdot \left(-\frac{1}{2}\right) = -3$

Vertical Component:

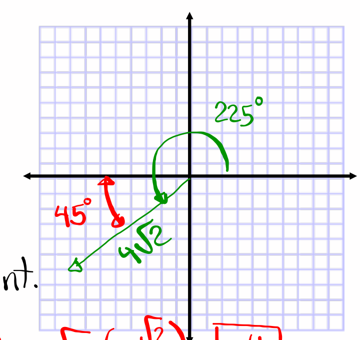
$|v|\sin\theta = 6 \cdot \sin 120° = 6 \cdot \sin 60° = 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$

$$v = \langle |v|\cos\theta, |v|\sin\theta \rangle = |v|\langle \cos\theta, \sin\theta \rangle$$
$$= 6\langle \cos 120°, \sin 120° \rangle$$
$$= \langle -3, 3\sqrt{3} \rangle$$

Jan 30-7:59 AM

$|u| = 4\sqrt{2}$, $\theta = 225°$

1) Draw u



2) Find horizontal Component.

$|u|\cos\theta = 4\sqrt{2} \cdot \cos 225° = 4\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) = \boxed{-4}$

3) Find Vertical Component

$|u|\sin\theta = 4\sqrt{2}\sin 225° = 4\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) = \boxed{-4}$

4) Express u using i & j notation.

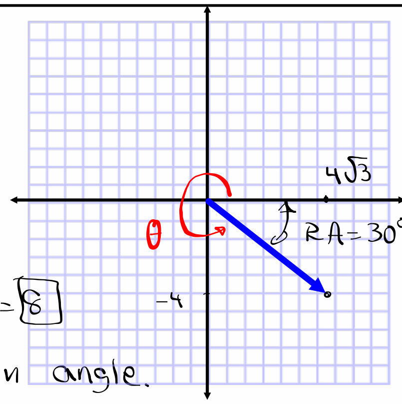
$u = |u|\cos\theta\, i + |u|\sin\theta\, j$

$\boxed{u = -4i - 4j}$

Jan 30-8:08 AM

$u = 4\sqrt{3}\, i - 4j$

1) Draw u



2) $|u| = \sqrt{(4\sqrt{3})^2 + (-4)^2}$
$= \sqrt{48 + 16} = \sqrt{64} = \boxed{8}$

3) Find its direction angle.

$v = \underbrace{|v|\cos\theta}_{4\sqrt{3}}\, i + \underbrace{|v|\sin\theta}_{-4}\, j$

$|v|\cos\theta = 4\sqrt{3}$ $\quad$ $|v|\sin\theta = -4$

$8\cos\theta = 4\sqrt{3}$ $\quad$ $8\sin\theta = -4$

$\cos\theta = \frac{\sqrt{3}}{2}$ $\quad$ $\sin\theta = \frac{-1}{2}$

R.A. $30°$

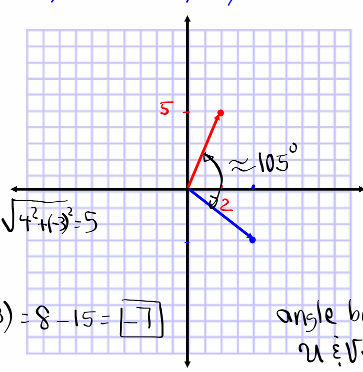
$\Rightarrow \theta = 360° - 30°$

$\boxed{\theta = 330°}$

Jan 30-8:15 AM

Given $\quad u = \langle 2, 5 \rangle \quad , \quad v = \langle 4, -3 \rangle$

1) Draw u & v



2) Find $|u|$ & $|v|$

$|u| = \sqrt{2^2 + 5^2} = \sqrt{29}$, $|v| = \sqrt{4^2 + (-3)^2} = 5$

3) Find $u \cdot v$

$u \cdot v = 2(4) + 5(-3) = 8 - 15 = \boxed{-7}$

angle between u & v

The dot product theorem: $u \cdot v = |u|\,|v| \cos\theta$

If we solve $\cos\theta = \dfrac{u \cdot v}{|u|\,|v|}$

For our example $\quad \cos\theta = \dfrac{-7}{\sqrt{29} \cdot 5} \Rightarrow \cos\theta = -.260$

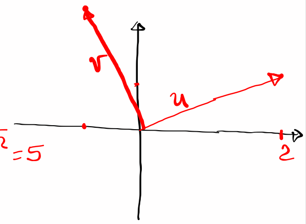
$\theta = \cos^{-1}(-.260)$

$\boxed{\theta \approx 105°}$

Jan 30-8:40 AM

$u=\langle 2,1\rangle$, $v=\langle -1,2\rangle$

1) Draw u & v



$|u|=\sqrt{2^2+1^2}=\sqrt{5}$, $|v|=\sqrt{(-1)^2+2^2}=5$

2) Find $|u|$ & $|v|$

3) Find $u\cdot v=2(-1)+1(2)=-2+2=\boxed{0}$

4) Find the angle between u & v.

$$\cos\theta=\frac{u\cdot v}{|u||v|}=\frac{0}{\sqrt{5}\sqrt{5}}=\frac{0}{5}=0 \rightarrow \theta=\cos^{-1}(0)$$

$$\theta=90^\circ$$

u & v are $\underbrace{\text{perpendicular}}_{\text{Orthogonal}}$ to each other.

If $u\cdot v=0$, then $u\perp v$.

Assuming u & v are not Zero Vectors.

Jan 30-8:49 AM

Are u & v orthogonal vectors?

1) $u=\langle 6,4\rangle$ $\quad v=\langle -2,3\rangle$

$u\cdot v=6(-2)+4(3)=-12+12=0$ **Yes**

2) $u=4i$, $v=-i+3j$

$u=\langle 4,0\rangle$ $\quad v=\langle -1,3\rangle$

$u\cdot v=4(-1)+0(3)=-4\neq 0$ **No**

Jan 30-8:56 AM

Find the angle between $u = 3i + 4j$ and $v = -2i - j$.

$$\cos\theta = \frac{u \cdot v}{|u|\,|v|}$$

$u = \langle 3, 4 \rangle$

$v = \langle -2, -1 \rangle$

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{|u|\,|v|}\right)$$

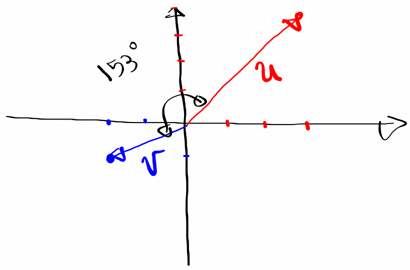
$u \cdot v = 3(-2) + 4(-1) = -6 - 4 = \boxed{-10}$

$|u| = \sqrt{3^2 + 4^2} = 5$

$|v| = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5}$

$$\theta = \cos^{-1}\left(\frac{-10}{5\sqrt{5}}\right)$$

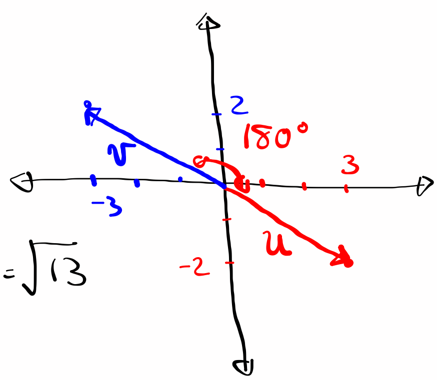
$$= \cos^{-1}\left(\frac{-2}{\sqrt{5}}\right) \approx 153^\circ$$



Jan 30-9:01 AM

$u = \langle 3, -2 \rangle$, $v = \langle -3, 2 \rangle$

1) Draw u & v.



2) Find $|u|$ & $|v| = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$

$|u| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$

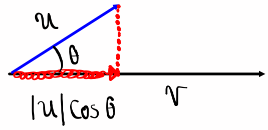
3) Find $u \cdot v = 3(-3) + (-2)(2) = -13$

4) Find the angle between u & v.

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{|u|\,|v|}\right) = \cos^{-1}\left(\frac{-13}{\sqrt{13}\sqrt{13}}\right) = \cos^{-1}\left(\frac{-13}{13}\right) = \cos^{-1}(-1)$$

$\boxed{\theta = 180^\circ}$

Jan 30-9:07 AM



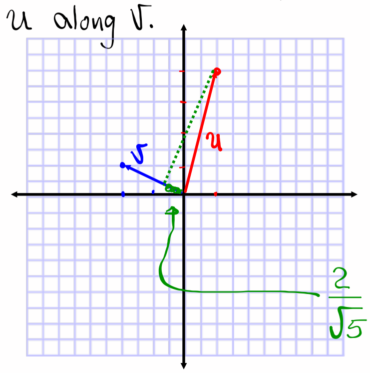
$$|u|\cos\theta = \frac{|v||u|\cos\theta}{|v|} = \frac{u\cdot v}{|v|}$$

$\underbrace{\text{Component of } u \text{ along } v}$

$u = \langle 1, 4\rangle$, $v = \langle -2, 1\rangle$

$$\frac{u\cdot v}{|v|} = \frac{2}{\sqrt{5}}$$

find component of u along v.

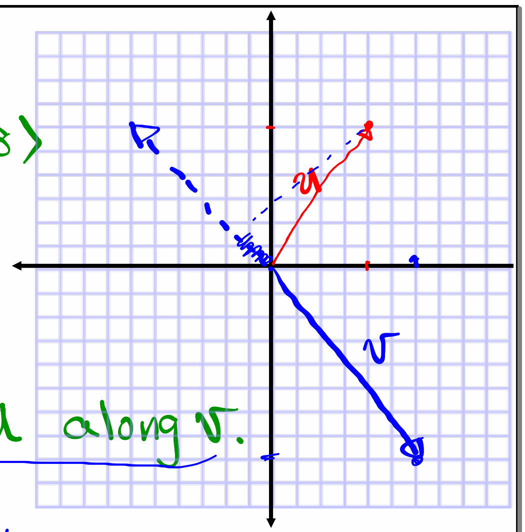


Jan 30-9:23 AM

Given

$u = \langle 4, 6\rangle$, $v = \langle 6, -8\rangle$

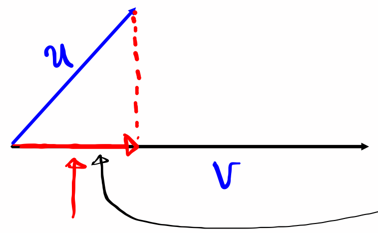
1) Draw u & v



2) find $\underbrace{\text{component of } u \text{ along } v}$.

$$\frac{u\cdot v}{|v|} = \frac{-24}{10} = \boxed{-2.4}$$

Jan 30-9:31 AM



$\text{Proj}_v u$ = Component of u onto v times a unit vector along v

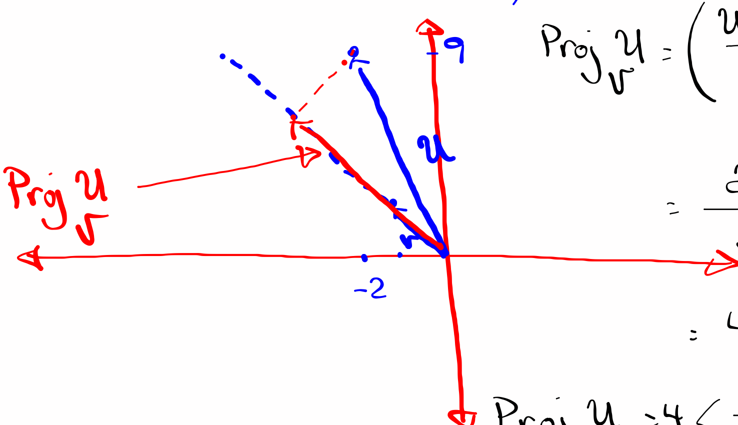
Projection of u onto v.

$$\text{Proj}_v u = \left(\frac{u \cdot v}{|v|}\right) \cdot \frac{1}{|v|} v = \left(\frac{u \cdot v}{|v|^2}\right) v$$

$$\text{Proj}_v u = \left(\frac{u \cdot v}{|v|^2}\right) v$$

Jan 30-9:38 AM

$u = \langle -2, 9 \rangle$, $v = \langle -1, 2 \rangle$



$$\text{Proj}_v u = \left(\frac{u \cdot v}{|v|}\right) \cdot \frac{1}{|v|} v$$

$$= \frac{20}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} v$$

$$= 4v$$

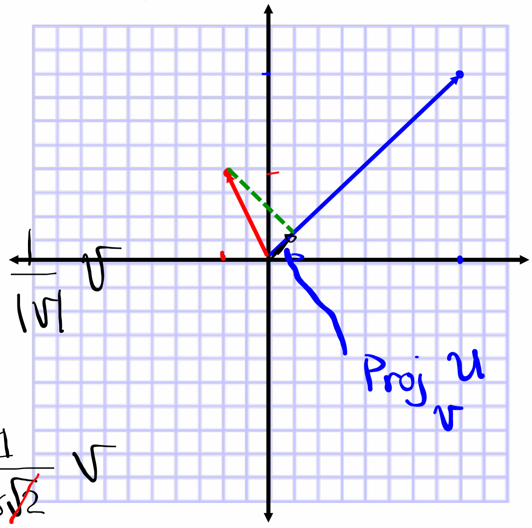
$$\text{Proj}_v u = 4\langle -1, 2 \rangle$$

$$= \langle -4, 8 \rangle$$

Jan 30-9:45 AM

$u = \langle -2, 4 \rangle \qquad v = \langle 8, 8 \rangle$

1) Draw u & v



2) $\text{Proj}_v u = \left(\dfrac{u \cdot v}{|v|}\right) \cdot \dfrac{1}{|v|} v$

$= \dfrac{\cancel{16}}{\cancel{8\sqrt{2}}} \cdot \dfrac{1}{8\cancel{\sqrt{2}}} v$

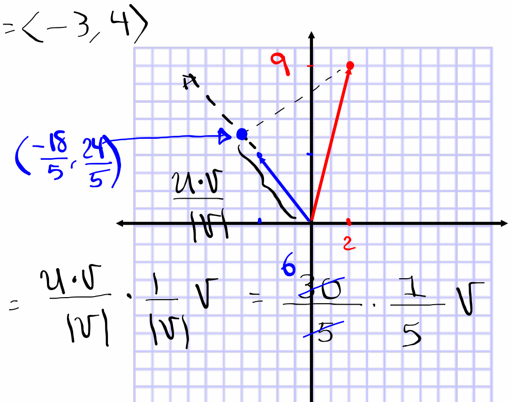
$= \dfrac{1}{8} v = \dfrac{1}{8}\langle 8, 8 \rangle = \langle 1, 1 \rangle$

Proj$_v u$

Jan 30-9:51 AM

$u = \langle 2, 9 \rangle, \quad v = \langle -3, 4 \rangle$

1) Draw u & v



2) Find $\text{Proj}_v u = \dfrac{u \cdot v}{|v|} \cdot \dfrac{1}{|v|} v = \dfrac{\overset{6}{\cancel{30}}}{\cancel{5}} \cdot \dfrac{1}{5} v$

$= \dfrac{6}{5} v$

$= \dfrac{6}{5}\langle -3, 4 \rangle$

$= \left\langle \dfrac{-18}{5}, \dfrac{24}{5} \right\rangle$

Vector u can be written as $u_1 + u_2$ where

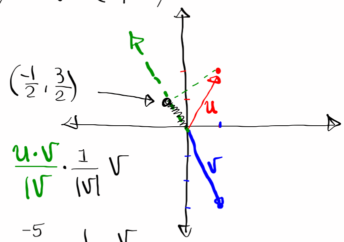
$u_1 = \text{Proj}_v u$

$u_2 = u - \text{Proj}_v u$

Jan 30-9:57 AM

Given $u = \langle 1,2\rangle$, $v = \langle 1,-3\rangle$

1) Draw u & v



2) find $\text{Proj}_v u = \frac{u\cdot v}{|v|}\cdot\frac{1}{|v|}v$

$= \frac{-5}{\sqrt{10}}\cdot\frac{1}{\sqrt{10}}v$

$= \frac{-5}{10}v = \frac{-1}{2}\langle 1,-3\rangle = \langle \frac{-1}{2}, \frac{3}{2}\rangle$

$u_1 = \text{Proj}_v u = \langle \frac{-1}{2}, \frac{3}{2}\rangle$

$u_2 = u - \text{Proj}_v u = \langle 1,2\rangle - \langle \frac{-1}{2}, \frac{3}{2}\rangle$

$= \langle 1 - \frac{-1}{2}, 2-\frac{3}{2}\rangle = \langle 1+\frac{1}{2}, 2-\frac{3}{2}\rangle$

$= \langle \frac{3}{2}, \frac{1}{2}\rangle$

$u = u_1 + u_2 = \langle \frac{-1}{2}, \frac{3}{2}\rangle + \langle \frac{3}{2}, \frac{1}{2}\rangle = \langle \frac{-1}{2}+\frac{3}{2}, \frac{3}{2}+\frac{1}{2}\rangle$

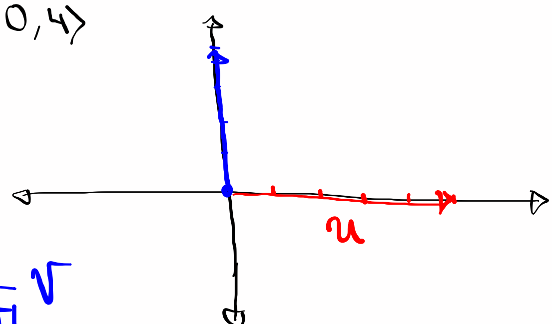
$= \langle \frac{2}{2}, \frac{4}{2}\rangle$

$= \langle 1, 2\rangle$

Jan 30-10:06 AM

$u = \langle 5,0\rangle \qquad v = \langle 0,4\rangle$

1) Draw u & v



2) $\text{Proj}_v u = \frac{u\cdot v}{|v|}\cdot\frac{1}{|v|}v$

$= \frac{0}{4}\cdot\frac{1}{4}v = 0\,v = \vec{0}$

SG 21 due at 7:00 in person Tuesday.

Jan 30-10:18 AM

Introduction to polar Coordinate system

1) Polar axis (x-axis , $x \geq 0$)

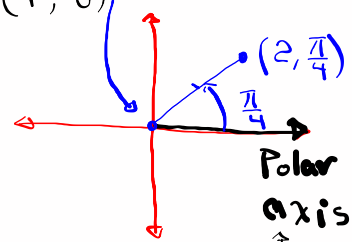
2) angle θ

3) distance from Pole (origin) r

Polar Coordinate (r, θ)

Plot $(2, \frac{\pi}{4})$

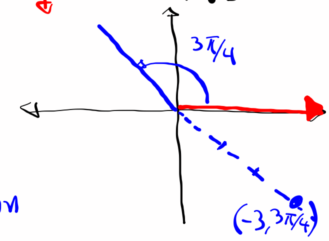
r θ



Plot $(-3, \frac{3\pi}{4})$

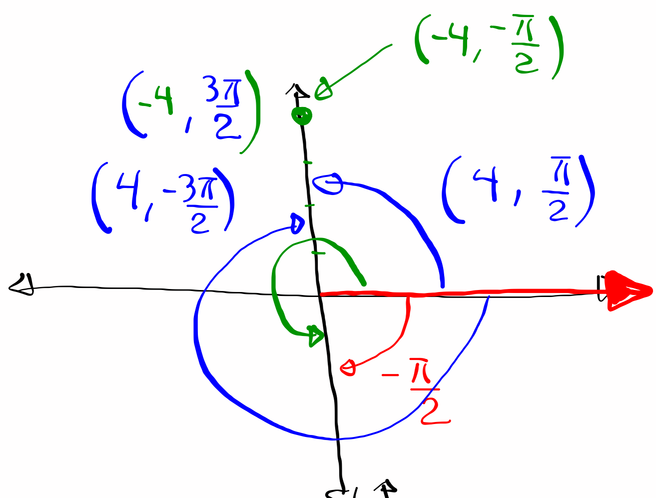
3 units from Pole

opposite direction



Jan 30-11:00 AM

Plot $(-4, -\frac{\pi}{2})$

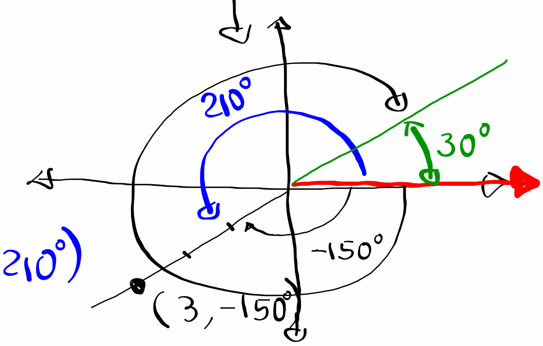


Plot $(3, -150^\circ)$

$(-3, 30^\circ)$

$(-3, -330^\circ)$

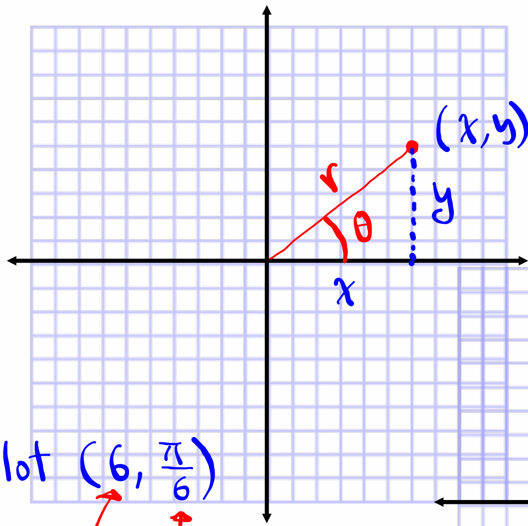
$(3, 210^\circ)$



Jan 30-11:06 AM

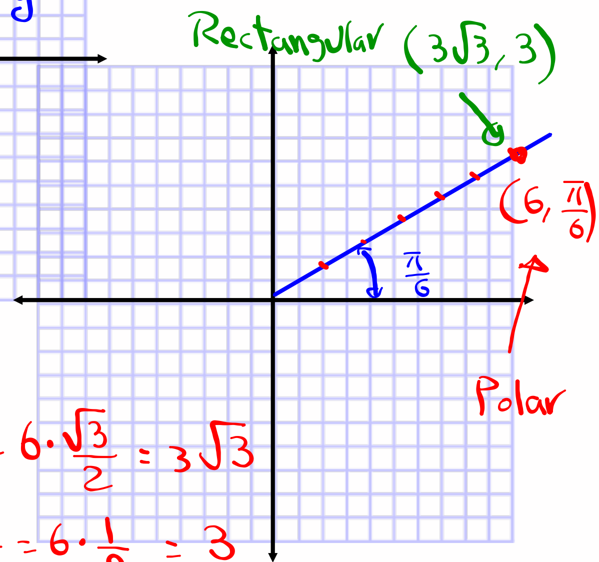
$\cos\theta = \frac{x}{r} \Rightarrow \boxed{x = r\cos\theta}$

$\sin\theta = \frac{y}{r} \Rightarrow \boxed{y = r\sin\theta}$



Plot $(6, \frac{\pi}{6})$

r θ



$x = r\cos\theta = 6\cos\frac{\pi}{6} = 6\cdot\frac{\sqrt{3}}{2} = 3\sqrt{3}$

$y = r\sin\theta = 6\cdot\sin\frac{\pi}{6} = 6\cdot\frac{1}{2} = 3$

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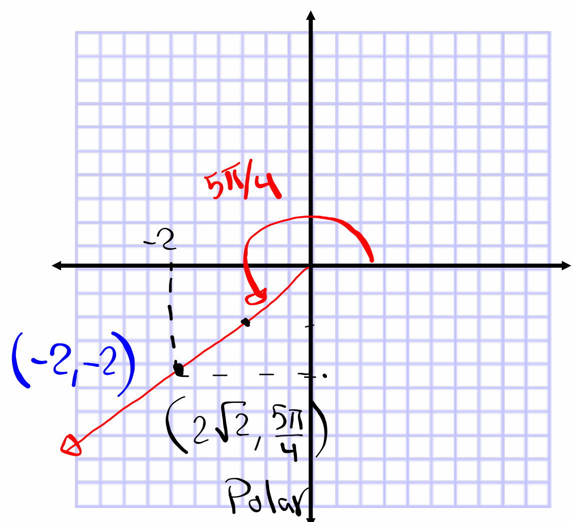
Plot $(2\sqrt{2}, \frac{5\pi}{4})$

r $\theta = \frac{5\pi}{4}$

$225°$



$x = r\cos\theta$

$= 2\sqrt{2}\cdot\cos\frac{5\pi}{4}$

$= 2\sqrt{2}\cdot -\frac{\sqrt{2}}{2} = \boxed{-2}$

$y = r\sin\theta = 2\sqrt{2}\cdot\sin\frac{5\pi}{4}$

$= \boxed{-2}$

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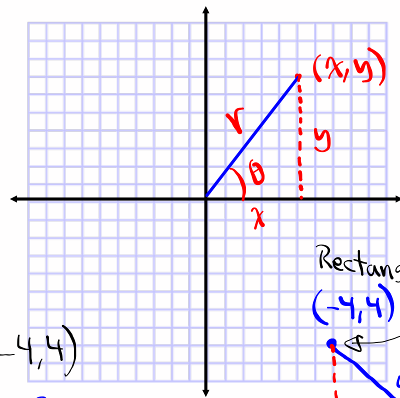
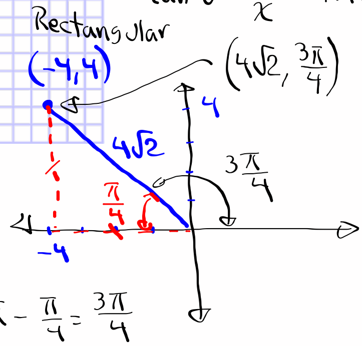


$r^2 = x^2 + y^2$

$x = r\cos\theta$

$y = r\sin\theta$

$\tan\theta = \dfrac{y}{x} \quad x \neq 0$



Plot $(-4, 4)$

$r^2 = x^2 + y^2$

$= (-4)^2 + 4^2$

$= 16 + 16$

$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$r^2 = 32$

$r = \sqrt{32} = \sqrt{16}\sqrt{2}$

$= \boxed{4\sqrt{2}}$

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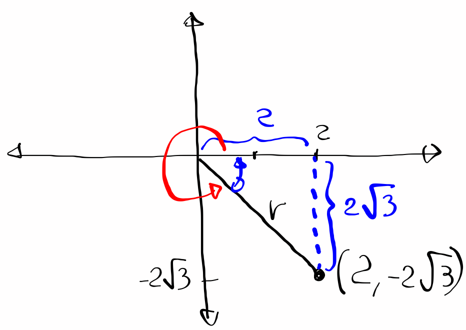
Consider the rectangular point $(2, -2\sqrt{3})$

1) Plot it.

2) find r



$r^2 = x^2 + y^2$

$= 2^2 + (-2\sqrt{3})^2$

$= 4 + 4 \cdot 3 = 16$

$r^2 = 16 \quad \boxed{r = 4}$

$\tan\theta = \dfrac{y}{x} = \dfrac{-2\sqrt{3}}{2} = -\sqrt{3}$

$\tan\theta = -\sqrt{3}$

RA. $60° = \frac{\pi}{3}$

$\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$(2, -2\sqrt{3})$ Rectangular

$\left(4, \frac{5\pi}{3}\right)$ Polar

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